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# ASYMPTOTICALLY AUTONOMOUS MULTIVALUED DIFFERENTIAL EQUATIONS(1)

BY

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ABSTRACT. The asymptotic behavior of solutions of the perturbed autonomous multivalued differential equation $x' \in F(x) + G(t, x)$ is examined in relation to the behavior of solutions of the autonomous equation $x' \in F(x)$ assuming that all solutions of the latter approach zero as $t$ approaches $\infty$.

For multivalued functions $F$ and $G$ whose values are nonempty subsets of $d$-dimensional Euclidean space, $R^d$, the generalized differential equation

$$x' \in F(x) + G(t, x) \tag{1}$$

is said to be asymptotically autonomous if $G(t, x)$ becomes small in some sense as $t \longrightarrow \infty$. The main result of this investigation establishes the relationship of the asymptotic behavior of solutions of (1) to that of solutions of the autonomous equation

$$x' \in F(x). \tag{2}$$

THEOREM 1. *Let $F$ be a positive-homogeneous upper semicontinuous mapping from $R^d$ (d-dimensional Euclidean space) to the nonempty, compact, convex subsets of $R^d$ such that all solutions of* (2) *approach zero as $t \longrightarrow \infty$. Let $G$ be a mapping from $R^{1+d}$ to the nonempty subsets of $R^d$ such that $G(t, \cdot) \longrightarrow 0$ as $t \longrightarrow \infty$ uniformly on nonempty compact subsets of $R^d$. If $\phi$ is a bounded solution of* (1) *on* $[0, \infty)$ *then* $\phi(t) \longrightarrow 0$ *as* $t \longrightarrow \infty$.

If $F$ and $G$ are single-valued functions, denoted by $f$ and $g$, respectively, the equations (1) and (2) are ordinary differential equations and the asymptotic behavior of the solutions is discussed, for example, by Strauss and Yorke. One of their results [7, p. 180] guarantees that all (classical) solutions of

$$x' = f(x) + g(t, x) \tag{3}$$

which are bounded on $[t_0, \infty)$ tend to zero as $t \longrightarrow \infty$ provided that $f$ and $g$ are

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continuous vector-valued functions, that all solutions of the unperturbed autonomous equation approach zero as $t \longrightarrow \infty$, and that $g(t, x)$ "mostly approaches zero". The last condition, which is defined in [7, p. 176] is satisfied, if for example, $g(t, \cdot)$ approaches zero as $t \longrightarrow \infty$ uniformly on compact subsets of $R^d$. Other treatments of asymptotically autonomous ordinary differential equations may be found in [1]–[4] and [6]–[10].

A perturbation-type result for generalized differential equations was developed by Lasota and Strauss [5, p. 169] as an aid in their investigation of autonomous ordinary differential equations. This result, tailored to suit the present context, is presented below.

LEMMA 2. *Let $F$ be a positive-homogeneous upper semicontinuous mapping from $R^d$ to the nonempty, compact convex subsets of $R^d$ such that every solution of* (2) *approaches zero as $t \longrightarrow \infty$. Then there exist $\epsilon > 0$ and $K > 1$ such that for $t_0 > 0$ and $x_0 \in R^d$ each solution of*

$$x' \in F(x) + \epsilon B(|x|), \qquad x(t_0) = x_0 \tag{4}$$

*can be continued to $+\infty$ and satisfies*

$$|x(t)| \leqslant K|x_0| \exp(-\epsilon(t - t_0)) \tag{5}$$

*for all $t \geqslant t_0$.*

A *solution* of (1) is an absolutely continuous $d$-vector valued function which satisfies (1) almost everywhere on some nondegenerate interval. For $\epsilon > 0$, $x \in R^n$, and $A \subset R^n$ denote the Euclidean norm of $x$ by $|x|$ and the norm of $A$ by $\|A\| = \sup\{|x|: x \in A\}$. The distance from $x$ to $A$ is defined by $d(x, A) = \inf\{|x - y|: y \in A\}$ and the $\epsilon$-neighborhood of $A$ is the set $N(A, \epsilon) = \{y \in R^n: d(y, A) < \epsilon\}$. The closed-origin-centered ball of radius $\epsilon$ is denoted by $B(\epsilon)$.

The multivalued mapping $H$ from $R^n$ to the nonempty compact subsets of $R^d$ is said to be *upper semicontinuous* if to each $\epsilon > 0$ and $x \in R^n$ there corresponds $\delta > 0$ such that $H(y) \subset N(H(x), \epsilon)$ provided $|x - y| < \delta$. The set-valued mapping $H$ defined on $R^n$ is said to be *positive-homogeneous* if $H(rx) = rH(x) = \{rz: z \in H(x)\}$ for all $x \in R^n$ and $r > 0$. The statement $H(t) \longrightarrow \infty$ means that to each $\epsilon > 0$ there corresponds $T > 0$ such that $H(t) \subset B(\epsilon)$ for all $t \geqslant T$; that is, $\|H(t)\| \longrightarrow 0$ as $t \longrightarrow \infty$.

A variation of Theorem 1, in which the perturbation term depends only on $t$, provides an approach to the proof of the main result.

THEOREM 3. *Let $F$ and $G$ satisfy the hypotheses of Theorem* 1 *and in addition assume that $G$ is independent of $x$. Then all solutions* (*not just the bounded solutions*) *of*

$$x' \in F(x) + G(t) \tag{6}$$

*on* $[0, \infty)$ *approach zero as* $t \longrightarrow \infty$.

The proof of this theorem is based on the observation that if $\phi$ is a solution of (6) for which $G(t) \subset \epsilon B(|\phi(t)|)$ for all $t \geqslant 0$ then $\phi(t) \longrightarrow 0$ as $t \longrightarrow \infty$ according to Lemma 2; whereas, if $G(t) \not\subset \epsilon B(|\phi(t)|)$ for all $t \geqslant 0$ then $\epsilon B(|\phi(t)|) \subset B(\|G(t)\|)$, and $\phi(t) \longrightarrow 0$ since $\|G(t)\| \longrightarrow 0$ as $t \longrightarrow \infty$.

PROOF OF THEOREM 3. Let $\epsilon$ and $K$ be as in Lemma 2 and let $\phi$ be a solution of (6) at least on $[0, \infty)$. Define the sets $I$ and $J$ by

$$I = \{t \geqslant 0\colon\ G(t) \subset \epsilon B(|\phi(t)|)\} \tag{7}$$

and

$$J = \{t \geqslant 0\colon\ G(t) \not\subset \epsilon B(|\phi(t)|)\}; \tag{8}$$

clearly $I \cup J = \{t \geqslant 0\}$ and $I \cap J = 0$. In the light of the previous remarks, it remains to be shown that $\phi(t) \longrightarrow 0$ as $t \longrightarrow \infty$ when both $I$ and $J$ are unbounded sets. Since the solution approaches zero on unbounded increasing sequences from $J$, it suffices to show that $\phi$ approaches zero along an arbitrary unbounded increasing sequence from $I$; let $\{t_k\}$ be such a sequence. For $k = 1, 2, 3, \ldots$, let $I_k$ denote the component (maximal connected subset) of $I$ which contains $t_k$ and let $d_k$ denote the length of this component. Let $s_k = \inf\{t \in I_k\}$ and assume, without loss of generality, that $s_1 > 1$; clearly, $s_k \nearrow \infty$ as $k \longrightarrow \infty$. The continuity of $\phi$ provides for each positive integer $k$ a corresponding $\delta_k < 1$ such that

$$|\phi(s_k) - \phi(t)| < 1/(2k) \quad \text{for } |t - s_k| < \delta_k; \tag{9}$$

in addition, if $d_k > 0$, choose $\delta_k < d_k$. Choose auxiliary sequences $\{\tau_k\} \subset J$ and $\{\tau_k^*\} \subset I$ such that $s_k - \delta_k \leqslant \tau_k \leqslant s_k$ and $s_k \leqslant \tau_k^* \leqslant s_k + \delta_k$ for each positive integer $k$; these selections guarantee that

$$|\phi(\tau_k) - \phi(s_k)| < 1/(2k) \tag{10}$$

and

$$|\phi(\tau_k^*) - \phi(s_k)| < 1/(2k). \tag{11}$$

Consequently, for $t \in I_k$, $\phi$ satisfies

$$|\phi(t)| \leqslant \begin{cases} |\phi(s_k)| + 1/(2k) & \text{for } t \leqslant \tau_k^*, \quad (12) \\ K|\phi(\tau_k^*)| \exp(-\epsilon(t - \tau_k^*)) & \text{for } t \geqslant \tau_k^*. \quad (13) \end{cases}$$

The estimate in (12) follows from (9) and the choice of $\tau_k^*$; whereas, the estimate in (13) follows from Lemma 2. These estimates can be modified by

(10) and (11) to obtain

$$|\phi(t)| \leqslant K(|\phi(\tau_k)| + 1/k) \quad \text{for } t \in I_k. \tag{14}$$

In particular, since $t_k \in I_k$ and $\tau_k \in J$, it follows that $\phi(\tau_k) \longrightarrow 0$ as $k \longrightarrow \infty$ which forces $\phi(t_k) \longrightarrow 0$ as $k \longrightarrow \infty$; thus $\phi(t) \longrightarrow 0$ as $t \longrightarrow \infty$, which concludes the proof.

The proof of Theorem 1 follows almost as an immediate consequence of Theorem 3.

PROOF OF THEOREM 1. Let $\phi$ be a bounded solution of (1) which is defined at least on $[0, \infty)$, and let $C$ denote a compact subset of $R^d$ which contains $\phi(t)$ for all $t \geqslant 0$. Define the multivalued function $H$ by

$$H(t) = \{y \in G(t, x): x \in C\}.$$

Clearly, $H(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and $\phi$ is a solution of

$$x' \in F(x) + G(t, x) \subset F(x) + H(t). \tag{15}$$

An application of Theorem 3 yields the desired results.

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